

# Magnetic Field Expansion Out of a Plane: Application to Inverse Cyclotron Muon Cooling

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## Abstract

In studies of the dynamics of charged particles in a cyclotron magnetic field, the specified field is generally  $B_z$  in the  $z = 0$  midplane where  $B_r$  and  $B_\theta$  are zero.  $B_r(r, \theta, z)$  and  $B_\theta(r, \theta, z)$  are simple to determine through a linear expansion which assumes that  $B_z$  is independent of  $z$ . But, an expansion to only first order may not be sufficient for orbit simulations at large  $z$ . This paper reviews the expansion of a specified  $B_z(r, \theta, z = 0)$  out of the  $z = 0$  midplane to arbitrary order, and shows simple examples worked out to 4th order.

**Keywords:** muon ionization cooling, neutrino factory, muon collider

## 1 Introduction

Many programs and software packages simulate orbits of charged particles in magnetic fields as noted in Table 1. Some use a single reference orbit and represent the magnetic field as a series of matrices, one for each magnetic element. Other programs allow more realistic spiral orbits by using magnetic field equations or interpolated magnetic field maps for point to point Runge-Kutta track integrators. Some programs such as COSY Infinity use exact analytical determination of magnetic field derivatives; others use numerical differentiation. An important preliminary step before simulation of fields produced by magnets is investigation of orbit dynamics in an idealized magnetic field which satisfies Maxwell's equations in vacuum to sufficient order. Magnetic fields for cyclotrons [1] are most conveniently expressed in cylindrical  $(r, \theta, z)$  coordinates in which the  $z$  direction is the cyclotron rotation axis. In many applications, an ideal field is specified in the cyclotron median plane which is the  $z = 0$  plane where  $B_r$  and  $B_\theta$  vanish. Determination of  $(B_r, B_\theta, B_z)$  for  $z \neq 0$  to sufficient order (beyond 1st order) is necessary for detailed orbit and dynamics studies [2]. Reduction of the phase space of a muon beam through ionization cooling in an inverse cyclotron is being researched [3] as part of the muon R&D for possible future facilities such as a neutrino factory and muon collider [4].

## 2 Expansion of Field Out of the Midplane

The importance of correct field expansion for reliable particle tracking simulation and beam dynamics analysis has been recognized, and the expansion around the reference orbit of a bent-solenoid geometry has been calculated [14]. Expansions to eighth order are done with the aid of Mathematica. A static magnetic field can be expanded out of a plane because it must satisfy the source-free Maxwell equations in vacuum,  $\nabla \cdot \mathbf{B} = 0$  and  $\nabla \times \mathbf{B} = 0$ . Only  $\partial B_x / \partial y = \partial B_y / \partial x$  constrains fields in the plane, so any function can be used for  $B_z$ . One gets the three field components out of the plane as follows

$$\frac{\partial B_z}{\partial z} = -\frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y} \quad \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} \quad \frac{\partial B_y}{\partial z} = \frac{\partial B_z}{\partial y}$$

The expansion of a magnetic field specified in a plane has been addressed by Zgoubi for cartesian coordinates [15]. Zgoubi uses Taylor expansion with  $\nabla \cdot \mathbf{B} = 0$  and  $\nabla \times \mathbf{B} = 0$ . Here we work out a solution in cylindrical coordinates to arbitrary order. Median plane antisymmetry is assumed so that

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Program	Tracks in Matter	Tracking Volume	Tracking Method	Reference
G4Beamline (GEANT4)	yes	unlimited	Point to Point	[5]
VORPAL	yes	unlimited	Point to Point	[6]
ICOOOL	yes	near a line	Point to Point	[7]
Methodical Accelerator Design (MAD-X)	no	near a line	Matrices or Points	[8]
SYNCH	no	near a line	Matrices or Points	[9]
Zgoubi	no	unlimited	Point to Point	[10]
COSY Infinity	yes	unlimited	Transfer Maps or Points	[11]
CYCLOPS and GOBLIN	no	near a plane	Point to Point	[12]
OptiM	yes	near a line	Matrix Elements	[13]

Table 1: Computer programs for tracking charged particles in magnetic fields and designing accelerators.

$$B_r(r, \theta, 0) = 0 \quad B_\theta(r, \theta, 0) = 0$$

$$B_r(r, \theta, z) = -B_r(r, \theta, -z) \quad B_\theta(r, \theta, z) = -B_\theta(r, \theta, -z) \quad B_z(r, \theta, z) = B_z(r, \theta, -z)$$

With  $B \equiv B_z(r, \theta, z = 0)$  the magnetic field is

$$B_r = z \frac{\partial B}{\partial r} - \frac{z^3}{6} \frac{\partial}{\partial r} \left( \frac{\partial^2 B}{\partial r^2} + \frac{1}{r} \frac{\partial B}{\partial r} + \frac{1}{r^2} \frac{\partial^2 B}{\partial \theta^2} \right) + \dots$$

$$B_\theta = \frac{z}{r} \frac{\partial B}{\partial \theta} - \frac{z^3}{6} \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial^2 B}{\partial r^2} + \frac{1}{r} \frac{\partial B}{\partial r} + \frac{1}{r^2} \frac{\partial^2 B}{\partial \theta^2} \right) + \dots$$

$$B_z = B - \frac{z^2}{2} \left( \frac{\partial^2 B}{\partial r^2} + \frac{1}{r} \frac{\partial B}{\partial r} + \frac{1}{r^2} \frac{\partial^2 B}{\partial \theta^2} \right) + \dots$$

which is

$$B_r = z \frac{\partial B}{\partial r} + \frac{\partial}{\partial r} \left[ \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^n B \right]$$

$$B_\theta = \frac{z}{r} \frac{\partial B}{\partial \theta} + \frac{1}{r} \frac{\partial}{\partial \theta} \left[ \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^n B \right]$$

$$B_z = B + \left[ \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^n B \right]$$

A rigorous proof of these expansions is provided in the final section of this paper.

In most implementations of simulating a magnetic field, the expansions of  $(B_r, B_\theta, B_z)$  cannot be expressed in closed form. In such cases, only a finite number of terms of the magnetic field expansion can be used. The order of the expansion is defined as the highest power of  $z$  in the summation. To odd (even) order  $m$ ,  $B_r$  and  $B_\theta$  have  $\frac{m+1}{2}$  ( $\frac{m}{2}$ ) terms while  $B_z$  has  $\frac{m+1}{2}$  ( $\frac{m+2}{2}$ ) terms. When  $\mathbf{B}$  is expanded to odd order,  $\nabla \times \mathbf{B} = 0$  and  $\nabla \cdot \mathbf{B} \neq 0$ ; when  $\mathbf{B}$  is expanded to even order,  $\nabla \cdot \mathbf{B} = 0$  and  $\nabla \times \mathbf{B} \neq 0$ . An even order expansion with  $\nabla \cdot \mathbf{B} = 0$  is needed to satisfy the Hamiltonian and allow a symplectic map [16].

### 3 Example Fields to 4th Order

This section shows some example midplane fields  $B_z(r, \theta, z = 0)$  expanded to 4th order to obtain  $B_r(r, \theta, z)$ ,  $B_\theta(r, \theta, z)$ , and  $B_z(r, \theta, z)$ . These example fields have fairly simple radial and azimuthal dependences and were checked with Mathematica. The 4th order expansion of a more complex field, one with a constant radial field index,  $k$  and spiral angle,  $\alpha$ , such as  $B_z(z = 0) = cr^k(1 - f \sin[N(\theta - \tan \alpha \ln(r/r_0))])$ , yields pages of output so that the use of a program such as Mathematica is necessary. Mathematica includes a command which produces output in **FORTRAN** or **C** syntax which can be copied directly into a routine which generates a grid of magnetic field points. Example Mathematica and **FORTRAN** files are presented in Appendices A and B.

#### 3.1 $B$ Only Dependent on Radius (No Sectors)

Is this case,  $B(r, \theta, z = 0)$  is  $B(r, z = 0)$  so that

$$B_r = z \frac{\partial B}{\partial r} - \frac{z^3}{6} \frac{\partial}{\partial r} \left( \frac{\partial^2 B}{\partial r^2} + \frac{1}{r} \frac{\partial B}{\partial r} \right)$$

$$B_\theta = 0$$

$$B_z = B(r, z = 0) - \frac{z^2}{2} \left( \frac{\partial^2 B}{\partial r^2} + \frac{1}{r} \frac{\partial B}{\partial r} \right) + \frac{z^4}{24} \left( \frac{\partial^4 B}{\partial r^4} + \frac{\partial^2}{\partial r^2} \left( \frac{1}{r} \frac{\partial B}{\partial r} \right) + \frac{1}{r} \frac{\partial^3 B}{\partial r^3} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial B}{\partial r} \right) \right)$$

An example of such a field is a non-sectored field in the middle of Helmholtz coils. At  $z = 0$ , the  $B_z$  is maximal at  $r = 0$  and decreases very slowly with radius so that  $\partial B / \partial r$  and higher derivatives are small, but still large enough to provide weak focusing in  $r$  and  $z$ .

#### 3.2 Azimuthal Field Sectors: No Spiral Angle nor Midplane Radial Dependence

For this field  $B(r, \theta, z = 0)$  is  $B(\theta, z = 0)$ : an example is  $B = B_0(1 - f \sin(N\theta))$  so that

$$B_r = \frac{1}{3} \left( \frac{z}{r} \right)^3 (f B_0 N^2 \sin(N\theta))$$

$$B_\theta = -\frac{z}{r} (f B_0 N \cos(N\theta)) - \frac{1}{6} \left( \frac{z}{r} \right)^3 (f B_0 N^3 \cos(N\theta))$$

$$B_z = B_0(1 - f \sin(N\theta)) - \frac{1}{2} \left( \frac{z}{r} \right)^2 (f B_0 N^2 \sin(N\theta)) + \frac{1}{24} \left( \frac{z}{r} \right)^4 (f B_0 N^2 \sin(N\theta))(4 - N^2)$$

A sectored field such as this provides stronger focusing in  $z$  than a non-sectored field.

#### 3.3 Azimuthal Field Sectors: Constant Spiral Angle, No Radial Field Index

For this field,  $B(r, \theta, z = 0)$  is  $B(\theta, z = 0)$ : a field with constant spiral angle  $\alpha$  is

$$B = B_0(1 - f \sin[N(\theta - \tan \alpha \ln(r/r_0))]) \text{ so that}$$

$$B_r = \frac{z}{r} (f B_0 N \tan \alpha) (\cos[N(\theta - \tan \alpha \ln(r/r_0))]) + \frac{1}{6} \left( \frac{z}{r} \right)^3 (f B_0 N^2 \sec^2 \alpha) (N \tan \alpha \cos[N(\theta - \tan \alpha \ln(r/r_0))] + 2 \sin[N(\theta - \tan \alpha \ln(r/r_0))])$$

$$B_\theta = -\frac{z}{r}(fB_0N)(\cos[N(\theta - \tan \alpha \ln(r/r_0))]) - \frac{1}{6}\left(\frac{z}{r}\right)^3 (fB_0N^3 \sec^2 \alpha)(\cos[N(\theta - \tan \alpha \ln(r/r_0))])$$

$$\begin{aligned} B_z &= B_0(1 - f \sin[N(\theta - \tan \alpha \ln(r/r_0))]) - \\ &\frac{1}{2}\left(\frac{z}{r}\right)^2 (fB_0N^2 \sec^2 \alpha)(\sin[N(\theta - \tan \alpha \ln(r/r_0))]) + \\ &\frac{1}{24}\left(\frac{z}{r}\right)^4 (fB_0N^2 \sec^2 \alpha)\{-N^2 \sec^2 \alpha \sin[N(\theta - \tan \alpha \ln(r/r_0))]\} + \\ &4(\sin[N(\theta - \tan \alpha \ln(r/r_0))] + N \tan \alpha \cos[N(\theta - \tan \alpha \ln(r/r_0))]) \} \end{aligned}$$

The spiraled magnetic field provides stronger focusing in  $z$  than the non-spiraled sector field. When  $\alpha = 0$ , the magnetic field components reduce to those of the previous azimuthally sector field with no spiral angle nor midplane radial dependence.

### 3.4 Azimuthal Field Sectors: No Spiral Angle, Constant Radial Field Index

$$k \equiv \frac{r}{B_z(z=0)} \frac{\partial B_z(z=0)}{\partial r}$$

For this field,  $B(r, \theta, z=0) = cr^k(1 - f \sin(N\theta))$ . Then

$$B_r = \left(\frac{z}{r}\right) (cr^k)k(1 - f \sin(N\theta)) - \frac{1}{6}\left(\frac{z}{r}\right)^3 (cr^k)(k-2)(fN^2 \sin(N\theta) + k^2(1 - f \sin(N\theta)))$$

$$B_\theta = -\left(\frac{z}{r}\right) (cr^k)fN \cos(N\theta) - \frac{1}{6}\left(\frac{z}{r}\right)^3 (cr^k)(fN \cos(N\theta))(N^2 - k^2)$$

$$\begin{aligned} B_z &= cr^k(1 - f \sin(N\theta)) - \\ &\frac{1}{2}\left(\frac{z}{r}\right)^2 (cr^k)(fN^2 \sin(N\theta) + k^2(1 - f \sin(N\theta))) + \\ &\frac{1}{24}\left(\frac{z}{r}\right)^4 (cr^k) [fN^2 \sin(N\theta)(2(k^2 - 2k + 2) - N^2) + (k-2)^2 k^2(1 - f \sin(N\theta))] \end{aligned}$$

## 4 Proof of Expansion Out of the Plane

Let

$$\mathbf{B}(r, \theta, z) = \hat{r}B_r(r, \theta, z) + \hat{\theta}B_\theta(r, \theta, z) + \hat{z}B_z(r, \theta, z) \equiv \mathbf{B}_\perp(r, \theta, z) + \hat{z}B_z(r, \theta, z),$$

$$\begin{aligned} \mathbf{B}_\perp(r, \theta, z) &= \sum_{k=0}^{\infty} \mathbf{A}_k(r, \theta) z^k, \\ B_z(r, \theta, z) &= \sum_{k=0}^{\infty} B_k(r, \theta) z^k, \end{aligned}$$

For any vector field  $\mathbf{V}$  and scalar field  $S$ ,

$$\begin{aligned} \vec{\nabla}_\perp \cdot \mathbf{V}_\perp &= \frac{1}{r} \frac{\partial}{\partial r}(rV_r) + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta}, \\ \vec{\nabla}_\perp S &= \hat{r} \frac{\partial S}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial S}{\partial \theta}, \\ \vec{\nabla}_\perp^2 S &\equiv \vec{\nabla}_\perp \cdot (\vec{\nabla}_\perp S) = \frac{\partial^2 S}{\partial r^2} + \frac{1}{r} \frac{\partial S}{\partial r} + \frac{1}{r^2} \frac{\partial^2 S}{\partial \theta^2} \end{aligned}$$

It is necessary and sufficient to show that the specification of  $B_0(r, \theta)$ ,  $\vec{\nabla} \cdot \mathbf{B} = 0$ ,  $\vec{\nabla} \times \mathbf{B} = 0 \Rightarrow$

$$\mathbf{A}_{2n+1}(r, \theta) = \frac{(-1)^n}{(2n+1)!} \vec{\nabla}_\perp \left( (\vec{\nabla}_\perp^2)^n B_0 \right) \text{ for } n \geq 1 \text{ and}$$

$$B_{2n}(r, \theta) = \frac{(-1)^n}{(2n)!} (\vec{\nabla}_\perp^2)^n B_0 \text{ for } n \geq 1.$$

Note that  $B$  was defined as  $B_0(r, \theta)$ .

$$\begin{aligned} \vec{\nabla} \cdot \mathbf{B} = 0 &\Rightarrow \\ \left( \vec{\nabla}_\perp + \hat{z} \frac{\partial}{\partial z} \right) \cdot (\mathbf{B}_\perp + \hat{z} B_z) &= \\ \vec{\nabla}_\perp \cdot \mathbf{B}_\perp + \frac{\partial B_z}{\partial z} &= \\ \sum_{k=0}^{\infty} (\vec{\nabla}_\perp \cdot \mathbf{A}_k) z^k + \sum_{k=1}^{\infty} k B_k z^{k-1} &= \\ \sum_{k=0}^{\infty} \left[ (\vec{\nabla}_\perp \cdot \mathbf{A}_k) + (k+1) B_{k+1} \right] z^k &= 0 \Rightarrow \\ (\vec{\nabla}_\perp \cdot \mathbf{A}_k) + (k+1) B_{k+1} &= 0 \Rightarrow \\ B_{k+1} &= -\frac{1}{k+1} (\vec{\nabla}_\perp \cdot \mathbf{A}_k) \text{ for } k = 0, 1, 2, \dots \end{aligned}$$

$$\begin{aligned} \vec{\nabla} \times \mathbf{B} = 0 &\Rightarrow \\ \left( \vec{\nabla}_\perp + \hat{z} \frac{\partial}{\partial z} \right) \times (\mathbf{B}_\perp + \hat{z} B_z) &= \\ (\vec{\nabla}_\perp \times \mathbf{B}_\perp) + \left( \vec{\nabla}_\perp \times (\hat{z} B_z) \right) + \left( \left( \hat{z} \frac{\partial}{\partial z} \right) \times \vec{\mathbf{B}}_\perp \right) &= \\ (\vec{\nabla}_\perp \times \mathbf{B}_\perp) - \hat{z} \times (\vec{\nabla}_\perp B_z) + \hat{z} \times \left( \frac{\partial \vec{\mathbf{B}}_\perp}{\partial z} \right) &= \\ (\vec{\nabla}_\perp \times \mathbf{B}_\perp) - \hat{z} \times \left( \vec{\nabla}_\perp B_z - \frac{\partial \vec{\mathbf{B}}_\perp}{\partial z} \right) &= 0. \\ (\vec{\nabla}_\perp \times \mathbf{B}_\perp) \text{ is parallel to } \hat{z} \text{ and } \hat{z} \times \left( \vec{\nabla}_\perp B_z - \frac{\partial \vec{\mathbf{B}}_\perp}{\partial z} \right) &\text{ is perpendicular to } \hat{z}. \\ \text{Therefore, } (\vec{\nabla}_\perp \times \mathbf{B}_\perp) = 0 \text{ and } \left( \vec{\nabla}_\perp B_z - \frac{\partial \vec{\mathbf{B}}_\perp}{\partial z} \right) &= 0. \\ \text{Then } \left( \vec{\nabla}_\perp B_z - \frac{\partial \vec{\mathbf{B}}_\perp}{\partial z} \right) = 0 &\Rightarrow \sum_{k=0}^{\infty} (\vec{\nabla}_\perp B_k) z^k - \sum_{k=1}^{\infty} k \mathbf{A}_k z^{k-1} = 0 \Rightarrow \\ \sum_{k=0}^{\infty} \left( (\vec{\nabla}_\perp B_k) - (k+1) \mathbf{A}_{k+1} \right) z^k &= 0 \Rightarrow \\ \mathbf{A}_{k+1} &= \frac{1}{k+1} (\vec{\nabla}_\perp B_k) \text{ for } k = 0, 1, 2, \dots \end{aligned}$$

$$\begin{aligned} B_{k+1} &= -\frac{1}{k+1} (\vec{\nabla}_\perp \cdot \mathbf{A}_k) \text{ and } \mathbf{A}_{k+1} = \frac{1}{k+1} (\vec{\nabla}_\perp B_k) \Rightarrow \\ B_{k+2} &= -\frac{1}{(k+2)(k+1)} \left( \vec{\nabla}_\perp \cdot (\vec{\nabla}_\perp B_k) \right) = -\frac{1}{(k+1)(k+2)} \vec{\nabla}_\perp^2 B_k \end{aligned}$$

$$\begin{aligned} k = 0 &\Rightarrow \\ \mathbf{A}_1 &= \vec{\nabla}_\perp B_0 \\ B_2 &= -\frac{1}{2} \vec{\nabla}_\perp^2 B_0 \end{aligned}$$

$$\begin{aligned} \mathbf{A}_0 &= 0 \\ B_1 &= 0 \end{aligned}$$

$$\begin{aligned} k = 1 &\Rightarrow \\ \mathbf{A}_2 &= \frac{1}{2} \vec{\nabla}_\perp B_1 = 0 \\ B_3 &= -\frac{1}{(3)(2)} \vec{\nabla}_\perp^2 B_1 = 0 \end{aligned}$$

$$\begin{aligned} k = 2 &\Rightarrow \\ \mathbf{A}_3 &= \frac{1}{3} \vec{\nabla}_\perp B_2 = -\frac{1}{(3)(2)} \vec{\nabla}_\perp \left( \vec{\nabla}_\perp^2 B_0 \right) = -\frac{1}{3!} \vec{\nabla}_\perp \left( \vec{\nabla}_\perp^2 B_0 \right) \\ B_4 &= -\frac{1}{(4)(3)} \vec{\nabla}_\perp^2 B_2 = \frac{1}{(4)(3)(2)} \vec{\nabla}_\perp^2 \left( \vec{\nabla}_\perp^2 B_0 \right) = \frac{1}{4!} \vec{\nabla}_\perp^2 \left( \vec{\nabla}_\perp^2 B_0 \right) \end{aligned}$$

Continuing recursively,

$$\begin{aligned} k = 2n - 1 &\Rightarrow \\ \mathbf{A}_{2n} &= 0 \text{ for } n \geq 1 \end{aligned}$$

$$B_{2n-1} = 0 \text{ for } n \geq 1$$

$$k = 2n \Rightarrow \mathbf{A}_{2n+1} = \frac{(-1)^n}{(2n+1)!} \vec{\nabla}_\perp \left( (\vec{\nabla}_\perp^2)^n B_0 \right) \text{ for } n \geq 1$$

$$B_{2n} = \frac{(-1)^n}{(2n)!} (\vec{\nabla}_\perp^2)^n B_0 \text{ for } n \geq 1$$

The proof is complete.

## 5 Summary

The procedure for expanding a specified  $B_z(r, \theta, z = 0)$  out of the  $z = 0$  midplane is described. Examples of expansions out to 4th order are included. Fig. 1 shows an example of a magnetic field.

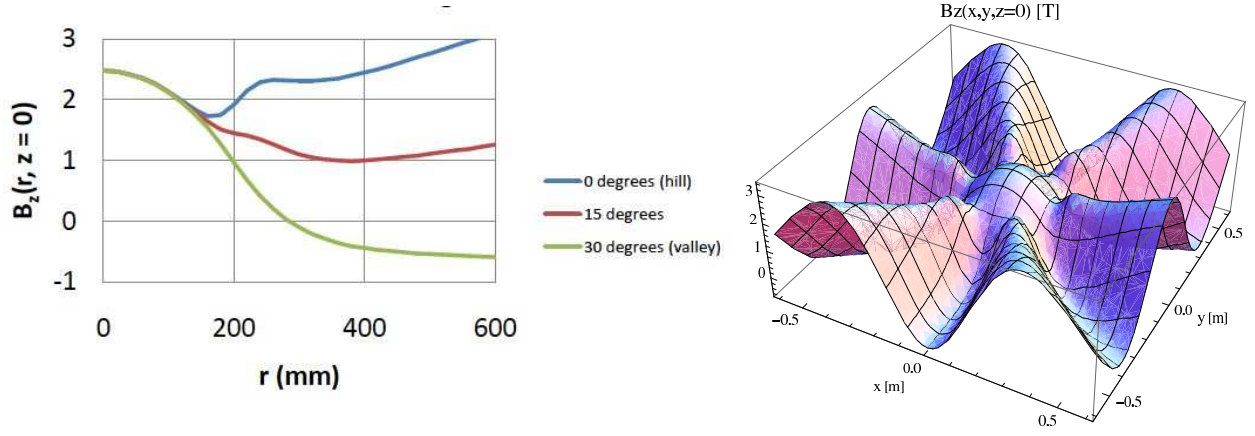


Figure 1:  $B_z(x, y, z = 0) = 1.77r^{0.6}(1 + \sqrt{2}\cos(6\theta))(1 + \tanh(25(r - 0.2)))/2$  including the outer 6-sector focus field which merges into the inner magnetic bottle.  $B_z(x, y, z = 0)$  ranges from -0.6 to 3.2 Tesla. The inner bottle field is generated by two  $r = 0.2$  m circular coils, located 0.2 m above and below the midplane. The  $B_z$  field at  $r = 0, z = 0$  is 2.4 Tesla.

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## A Appendix: Mathematica Example

An example of a Mathematica file which evaluates the coefficients of  $B_z(r, \theta, z = 0) = cr^k(1 + f \cos(N\theta))$  up to 4th order is shown. This  $B_z(r, \theta, z = 0)$  is simpler than the field of Fig. 1 so that the example file and output are reasonably short and clear. The example file also formats the expressions of the coefficients to FORTRAN. The next appendix is an example FORTRAN routine which uses the output FORTRAN expressions from this file to generate a grid of magnetic field points used by a G4Beamline input file (Appendix C). This Mathematica file shows the output of each Mathematica command indented.

```
(* Set b as Bz(z = 0) with (r, t) as radius and theta. *)
b[r_,t_] = (c r^k)(1+f Cos[n ( t)])

      c r^k (1+f Cos[n t])

(* Set trans as d^2b/dr^2 + (1/r)db/dr + (1/r^2)d^2b/dt^2. *)
trans[r_,t_] = D[b[r,t],{r,2}] + (1/r)D[b[r,t],{r,1}] + (1/r^2)D[b[r,t],{t,2}]

      -c f n^2 r^(-2+k) Cos[n t] + c k r^(-2+k) (1+f Cos[n t]) + c (-1+k) k
      r^(-2+k) (1+f Cos[n t])

(* FullSimplify does an algebraic simplification *)
FullSimplify[%]

      c r^(-2+k) (k^2+f (k-n) (k+n) Cos[n t])

(* trans2 is a double application of the derivatives of trans. *)
trans2[r_,t_] = D[trans[r,t],{r,2}] + (1/r)D[trans[r,t],{r,1}] +
(1/r^2)D[trans[r,t],{t,2}]

      -c f (-3+k) (-2+k) n^2 r^(-4+k) Cos[n t] + c (-3+k) (-2+k) k r^(-4+k)
      (1+f Cos[n t]) + c (-3+k) (-2+k) (-1+k) k r^(-4+k) (1+f Cos[n t]) +
```



```

(-c f k n^2 r^(-2+k) Cos[n t]-c f (-1+k) k n^2 r^(-2+k) Cos[n t]+
c f n^4 r^(-2+k) Cos[n t])/r^2+1/r (-c f (-2+k) n^2 r^(-3+k) Cos[n t]+
c (-2+k) k r^(-3+k) (1+f Cos[n t]))+c (-2+k) (-1+k) k r^(-3+k)
(1+f Cos[n t]))

FullSimplify[%]

c r^(-4+k) ((-2+k)^2 k^2+f (-2+k-n) (k-n) (-2+k+n) (k+n) Cos[n t])

(* Set field components, apply simplifications, convert to *)
(* FORTRAN format. *)

(* Use o1, o2, o3, and o4 as switches to set order of expansion. *)
(* For 2nd order, set o1 = 1, o2 = 1, o3 = o4 = 0. *)

(* Set the 0th, 2nd, and 4th order terms of Bz(z). *)
bz0[r_,t_] = b[r,t]

c r^k (1+f Cos[n t])

FullSimplify[%]

c r^k (1+f Cos[n t])

FortranForm[%]

c*r**k*(1 + f*Cos(n*t))

bz2[r_,t_] = -(o2)(z^2/2)trans[r,t]

-(1/2) o2 z^2 (-c f n^2 r^(-2+k) Cos[n t]+c k r^(-2+k) (1+f Cos[n t]))+
c (-1+k) k r^(-2+k) (1+f Cos[n t]))

FullSimplify[%]

-(1/2) c o2 r^(-2+k) z^2 (k^2+f (k-n) (k+n) Cos[n t])

FortranForm[%]

-(c*o2*r**(-2 + k)*z**2*(k**2 + f*(k - n)*(k + n)*Cos(n*t)))/2.

bz4[r_,t_] = (o4)(z^4/24)trans2[r,t]

1/24 o4 z^4 (-c f (-3+k) (-2+k) n^2 r^(-4+k) Cos[n t]+
c (-3+k) (-2+k) k r^(-4+k) (1+f Cos[n t]))+
c (-3+k) (-2+k) (-1+k) k r^(-4+k) (1+f Cos[n t]))+
(-c f k n^2 r^(-2+k) Cos[n t]-c f (-1+k) k n^2 r^(-2+k) Cos[n t]+
c f n^4 r^(-2+k) Cos[n t])/r^2+1/r (-c f (-2+k) n^2 r^(-3+k) Cos[n t]+
c (-2+k) k r^(-3+k) (1+f Cos[n t]))+
c (-2+k) (-1+k) k r^(-3+k) (1+f Cos[n t]))

FullSimplify[%]

1/24 c o4 r^(-4+k) z^4 ((-2+k)^2 k^2+f (-2+k-n) (k-n) (-2+k+n) (k+n)

```

```

Cos[n t])

FortranForm[%]

(c*o4*r**(-4 + k)*z**4*
- ((-2 + k)**2*k**2 +
- f*(-2 + k - n)*(k - n)*(-2 + k + n)*(k + n)*Cos(n*t)))/24.

(* Set the 1st and 3rd order terms of Br(z). *)
br1[r,t] = (o1)z D[b[r,t],{r,1}]

c k o1 r^(-1+k) z (1+f Cos[n t])

FullSimplify[%]

c k o1 r^(-1+k) z (1+f Cos[n t])

FortranForm[%]

c*k*o1*r**(-1 + k)*z*(1 + f*Cos(n*t))

br3[r,t] = -(o3)(z^3/6)D[trans[r,t],{r,1}]

-(1/6) o3 z^3 (-c f (-2+k) n^2 r^(-3+k) Cos[n t]+
c (-2+k) k r^(-3+k) (1+f Cos[n t]))+
c (-2+k) (-1+k) k r^(-3+k) (1+f Cos[n t]))

FullSimplify[%]

-(1/6) c (-2+k) o3 r^(-3+k) z^3 (k^2+f (k-n) (k+n) Cos[n t])

FortranForm[%]

-(c*(-2 + k)*o3*r**(-3 + k)*z**3*
- (k**2 + f*(k - n)*(k + n)*Cos(n*t)))/6.

(* Set the 1st and 3rd order terms of Btheta(z). *)
bt1[r,t] = (o1)z (1/r) D[b[r,t],{t,1}]

-c f n o1 r^(-1+k) z Sin[n t]

FullSimplify[%]

-c f n o1 r^(-1+k) z Sin[n t]

FortranForm[%]

-(c*f*n*o1*r**(-1 + k)*z*Sin(n*t))

bt3[r,t] = -(o3)(z^3/6)(1/r)D[trans[r,t],{t,1}]

-1/(6 r) o3 z^3 (-c f k n r^(-2+k) Sin[n t]-
c f (-1+k) k n r^(-2+k) Sin[n t]+c f n^3 r^(-2+k) Sin[n t])

```

FullSimplify[%]

$\frac{1}{6} c f (k-n) n (k+n) o_3 r^{(-3+k)} z^3 \sin[n t]$

FortranForm[%]

$(c*f*(k - n)*n*(k + n)*o_3*r**(-3 + k)*z**3*\sin(n*t))/6.$

## B Appendix: FORTRAN Example

This program takes the output expressions of  $B_z(z)$ ,  $B_r(z)$ , and  $B_\theta(z)$  from Mathematica and prints out a grid of magnetic field values recast as  $B_x$ ,  $B_y$ , and  $B_z$ . The output file generated by this routine is fort.71. With further formatting described in the initial comments of the routine, the output file can be used by a G4Beamline input file to generate a magnetic field for an anticyclotron simulation. In this example, the field componenets are expanded to 4th order in  $z$ .

```
PROGRAM FOURTH_ORDER_ARXIV_EXAMPLE

C
C Prints out grid of (Bx, By, Bz) in (x, y, z) for a file suitable for the
C G4Beamline command 'fieldmap'.
C
C The output of this program is fort.71.
C
C Distances in the fort.71 file are in mm., and magnetic fields are
C in tesla. (Distances in this FORTRAN routine are in meters.)
C
C The following must be added to fort.71 for the file to work in G4Beamline:
C
C grid X0=[smallest x] Y0=[smallest y] Z0=[smallest z] nX=[number of x points]
C nY=[number of y points] nZ=[number of z points] dX=[dist. between x points]
C dY=[dist. between y points] dZ=[dist. between z points]
C data
C

IMPLICIT NONE

INTEGER I,J,L

REAL*8 x,y,z,r,t
REAL*8 bz0,bz2,bz4,br1,br3,bt1,bt3
REAL*8 bx,by,bz,br,bt

REAL*8 PI,E
REAL*8 b,c,k,s,f,n

REAL*8 o1,o2,o3,o4

E = 2.718281828d+0
PI = ACOS(-1.0d+0)

C Bz(z=0) = (c*r^k)*(1+f*cos(n(t)))
```

```

C Bz(z = 0) parameters
  b = 25.0d+0
  c = 1.77d+0
  k = 0.6d+0
  s = (PI/180.0d+0)*0.0d+0
  f = sqrt(2.0d+0)
  n = 6.0d+0

C o1 through o4 set the order of the expansion in z of Bz(z=0)
C 4th order:  o4 = o3 = o2 = o1 = 1
C 3rd order:  o4 = 0, o3 = o2 = o1 = 1
C 2nd order:  o4 = o3 = 0, o2 = o1 = 1
C 1st order:  o4 = o3 = o2 = 0, o1 = 1
C 0th order:  o4 = o3 = o2 = o1 = 0

  o1 = 1.0d+0
  o2 = 1.0d+0
  o3 = 1.0d+0
  o4 = 1.0d+0

C Loop over x, y, z
  DO 100 I = 1, 201
    DO 200 J = 1, 201
      DO 300 L = 1, 61

C x, y, z, r in meters
C t in radians
C br, bt, bz in Tesla

      x = -1.01d+0 + 0.01d+0*I
      y = -1.01d+0 + 0.01d+0*J
      z = -0.31d+0 + 0.01d+0*L

      r = sqrt(x*x+y*y)

      t = atan2(y,x)

C br, bt, bz formula determined through and copied and pasted from Mathematica

C bz terms up to 4th order
  bz0 = c*r**k*(1 + f*cos(n*t))

  bz2 = -(c*o2*r**(-2 + k)*z**2*(k**2 +
-      f*(k - n)*(k + n)*cos(n*t)))/2.

  bz4 = (c*o4*r**(-4 + k)*z**4*
-      ((-2 + k)**2*k**2 +
-      f*(-2 + k - n)*(k - n)*(-2 + k + n)*(k + n)*cos(n*t)))/24.

C br terms up to 4th order
  br1 = c*k*o1*r**(-1 + k)*z*(1 + f*cos(n*t))

```

```

        br3 = -(c*(-2 + k)*o3*r**(-3 + k)*z**3*
-        (k**2 + f*(k - n)*(k + n)*Cos(n*t)))/6.

C  bt terms up to 4th order
        bt1 = -(c*f*n*o1*r**(-1 + k)*z*Sin(n*t))

        bt3 = (c*f*(k - n)*n*(k + n)*o3*r**(-3 + k)*z**3*Sin(n*t))/6.

C  Add each order to get bz, br, bt
        bz = bz0 + bz2 + bz4

        br = br1 + br3

        bt = bt1 + bt3

C  Transform br, bt to bx, by
        bx=br*COS(t)-bt*SIN(t)

        by=br*SIN(t)+bt*COS(t)

        WRITE(71,174) 1000*x,1000*y,1000*z,bx,by,bz,0.0

174      FORMAT(F8.1,"",F8.1,"",F8.1,"",F9.3,"",F9.3,"",F9.3,
1        ", 0.0, 0.0, ",F7.3)

300      CONTINUE
200      CONTINUE
100      CONTINUE

        END

```

Here are the first 10 lines in the fort.71 output file.

```

head fort.71
-1000.0, -1000.0, -300.0, -3.318, 3.712, 2.162, 0.0, 0.0, 0.000
-1000.0, -1000.0, -290.0, -3.160, 3.542, 2.163, 0.0, 0.0, 0.000
-1000.0, -1000.0, -280.0, -3.007, 3.375, 2.164, 0.0, 0.0, 0.000
-1000.0, -1000.0, -270.0, -2.859, 3.214, 2.165, 0.0, 0.0, 0.000
-1000.0, -1000.0, -260.0, -2.716, 3.057, 2.166, 0.0, 0.0, 0.000
-1000.0, -1000.0, -250.0, -2.576, 2.904, 2.167, 0.0, 0.0, 0.000
-1000.0, -1000.0, -240.0, -2.441, 2.756, 2.168, 0.0, 0.0, 0.000
-1000.0, -1000.0, -230.0, -2.310, 2.611, 2.169, 0.0, 0.0, 0.000
-1000.0, -1000.0, -220.0, -2.182, 2.471, 2.170, 0.0, 0.0, 0.000
-1000.0, -1000.0, -210.0, -2.058, 2.334, 2.171, 0.0, 0.0, 0.000

```

With a grid spacing of 10 *mm* ranging from  $-1000\text{ mm} < (x,y) < 1000\text{ mm}$  and  $-300\text{ mm} < z < 300\text{ mm}$ , the following lines need to be placed at the start of `fort.71` to match the syntax required by the G4Beamline command `fieldmap`

```

param normB=1.0000 normE=0.0000
grid X0=-1000 Y0=-1000 Z0=-300 nX=201 nY=201 nZ=61 dX=10 dY=10 dZ=10
data

```

## C Appendix: G4Beamline Example

An example G4Beamline [5] input file of positive muons being decelerated in an anticyclotron is shown. The file `test_bfield_6sector_rk06_0d_2nd_order_cos.example.dat` as specified in the `fieldmap field_grid file=test_bfield_6sector_rk06_0d_2nd_order_cos.example.dat` command is needed to generate the outer sectorized focusing field. The inner magnetic bottle is generated by the field from two current carrying coils. Muons lose energy by passing through  $1.0^\circ$  lithium hydride wedges. The uniform  $1.0^\circ$  wedge angle in this example file is simpler than the moderator configuration used for the anticyclotron. The anticyclotron includes an inner cylinder of 0.1 bar helium and six LiH wedges whose thickness decreases adiabatically from 4.2 mm at  $r = 550$  mm to 0.008 mm at  $r = 65$  mm. The output is a root file, `AllTracks.root`, which shows all the kinematic information of each muon along its orbit, and which can be opened by `Historoot`.

```
* example G4Beamline file for anticyclotron simulation, April 11, 2011
*
# use TJR recommended physics routines
physics QGSP_BERT doStochastics=1 disable=Decay

# reference particle with no stochastic processes
particlecolor reference=1,1,1
reference referenceMomentum=180 particle=mu+ \
    beamX=0.0 beamY=500.0 beamZ=0.0 meanXp=0.0 meanYp=0.0 rotation=X90,Z90

# simulate 3 muons with stochastic processes (since doStochastics=1)
beam gaussian particle=mu+ nEvents=3 \
beamX=0.0 beamY=500.0 beamZ=0.0 meanXp=0.0 meanYp=0.0 \
sigmaX=0 sigmaY=0 sigmaXp=0 sigmaYp=0 \
meanMomentum=180.00 sigmaP=0 meanT=0 sigmaT=0 rotation=X90,Z90

# keep only positive muons
trackcuts keep=mu+ maxTime=200000.0

# output file for Historoot showing kinematic information for orbits
trace nTrace=3 format=root filename=AllTracks.root oneNTuple=1 primaryOnly=1

# output file for Historoot showing kinematic information when particles are lost
beamlossntuple loss_ntuple format=ascii

param maxStep=5. SteppingVerbose=1

# lithium hydride, density (0.82 g/cm^3) from Wikipedia
material lih z=4 a=8 density=0.82 state=s

# Define LiH wedges, each with 1 degree angle
tubs gascylinder_1 innerRadius=2 outerRadius=600 length=200 \
    initialPhi=30-0.5 finalPhi=30+0.5 color=1,0,0 material=lih
tubs gascylinder_2 innerRadius=2 outerRadius=600 length=200 \
    initialPhi=90-0.5 finalPhi=90+0.5 color=1,0,0 material=lih
tubs gascylinder_3 innerRadius=2 outerRadius=600 length=200 \
    initialPhi=150-0.5 finalPhi=150+0.5 color=1,0,0 material=lih
tubs gascylinder_4 innerRadius=2 outerRadius=600 length=200 \
    initialPhi=210-0.5 finalPhi=210+0.5 color=1,0,0 material=lih
tubs gascylinder_5 innerRadius=2 outerRadius=600 length=200 \
    initialPhi=270-0.5 finalPhi=270+0.5 color=1,0,0 material=lih
tubs gascylinder_6 innerRadius=2 outerRadius=600 length=200 \
    initialPhi=330-0.5 finalPhi=330+0.5 color=1,0,0 material=lih
```

```

# Define coils of inner magnetic bottle
coil coil1 \
innerRadius=195 \
outerRadius=205 \
length=50 \
material=Vacuum \
filename=coil1.dat

coil coil2 \
innerRadius=195 \
outerRadius=205 \
length=50 \
material=Vacuum \
filename=coil2.dat

# Set currents for magnetic bottle coils which provide inner bottle magnetic field
solenoid c1 \
    coilName=coil1 \
    current=(0.2)*10803.6 \
    color=0.3,1,0 \
    alternate=0

solenoid c2 \
    coilName=coil1 \
    current=(0.2)*10803.6 \
    color=0.3,1,0 \
    alternate=0

# Grid of magnetic field points for outer sector field
fieldmap field_grid file=test_bfield_no_poly_bottle_6sector_rk06_0d_2nd_order_beq25_cos.dat

# Place wedges, coils, and magnetic field map grid
place gascylinder_1 z=0
place gascylinder_2 z=0
place gascylinder_3 z=0
place gascylinder_4 z=0
place gascylinder_5 z=0
place gascylinder_6 z=0

place c1 x=0 y=0 z=200
place c2 x=0 y=0 z=-200

place field_grid x=0 y=0 z=0

```

Fig. 2 shows a `Historoot` visualization of the orbits of three muons in this G4Beamline example simulation. The muons stop in about 125 *ns*.

## D Appendix: ICOOL Example

Another program which can simulate an anticyclotron with lithium hydride wedges is ICOOL [7]. The coordinate system of ICOOL is an  $(x, y, z)$  coordinate system in which the  $z$  coordinate is along the beamline. This can be converted to an  $(r, \theta, z)$  system by incorporating a bent solenoid so that  $x$  becomes  $-r$ ,  $y$  becomes

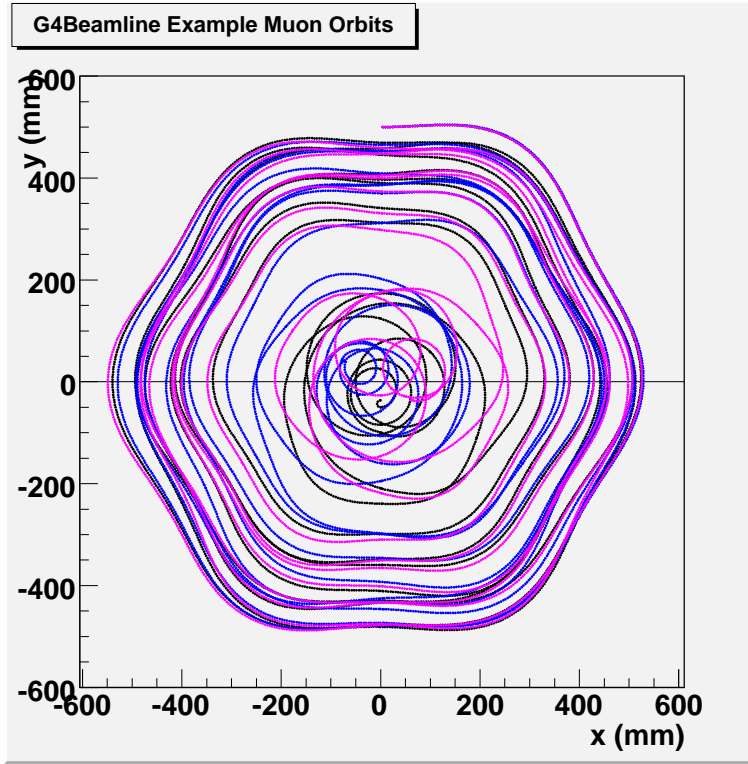


Figure 2: Orbits of three color coded muons in an G4Beamline simulated anticyclotron. This output is generated by **Historoot**. The orbits show the transition between the outer sectored field and the inner azimuthally symmetric magnetic bottle.

$z$ , and the reference path coordinate,  $z$ , becomes  $\theta$ . ICOOL does not have the visualization capabilities of G4Beamline so software to check element placement, orbits, and dynamics must be added. ICOOL can do its own magnetic field expansion [14] to check an orbit at a particular radius. This can provide a cross check of G4Beamline.

ICOOL can be run on UNIX, PC, and Macintosh platforms. The command file needs to be called **for001.dat**. In this particular application of ICOOL, two more input files are used: **for003.dat** which defines each particle of the beam and **for056.dat** which defines the magnetic field. The output file in the example simulation is **for009.dat** which lists the kinematic information of each particle. Postprocessing of this file is needed to visualize the orbits and to evaluate the dynamics.

Information about ICOOL including a User's Guide and Reference Manual can be found at

<https://pubweb.bnl.gov/~fernow/icool/v326/>

The next sections show example input files which generate an anticyclotron simulation when the executable **icool** from

<https://pubweb.bnl.gov/~fernow/icool/v326/icool.exe>

is run.

## D.1 input for001.dat command file

This command file uses a bent solenoid magnetic field to simulate a cyclotron field. The **BSOL** command sets the initial muon momentum to  $0.18 \text{ GeV}/c$ , the inverse radius of the bent solenoid to  $2.0 \text{ m}^{-1}$ , and reads the amplitude, period, and initial offset of the sinusoidal vertical magnetic field ( $B_y(z)$  where  $y$  is vertical and  $z$  is along the reference path) from the file **fort.56**. In this ICOOL example, muons lose energy by passing through  $1.0^\circ$  lithium hydride wedges. The magnetic field in this example does not vary with radius



is in the G4Beamline example of Appendix C. The geometry of the anticyclotron is specified by **sregions**. One **sregion** of the lithium hydride wedge is followed by 10 **sregions** of vacuum, and the sequence of 11 **sregions** is repeated 600 times. The total length of the 11 **sregions** is 0.523598775 *m* which corresponds to 1/6 of the circumference of a 0.5 *m* circle. This arrangement is set for a 6-sector anticyclotron with a bent solenoid radius of 0.5 *m*.

```
test ring 180 MeV (H5b)
```

```
$cont npart=200 nsections=1 varstep=.false. nprnt=-3 prlevel=1
ntuple=.false. rtuple=.false. rtuplen=1 output1=.true. phasemodel=3
fsav=.false. fsavset=.true. izfile=140 bgen=.false. $
```

```
$bmt $
1 2 1. 1 !ntyp typ(1e 2mu 3pi 4K 5p) frac dist(1=gauss 2=ring)
-0.010 0.019 0. 0. 0. 0.203 !means: x y z px py pz
0.0425 0.0425 .18 .030 .030 0.024 !sigmas: "
1 !correlations
2 .19 .40 0 !palmer GeV/A^2 beta
```

```
$ints ldecay=.false. declv=1 ldedx=.true. lstrag=.true. lscatter=.true.
delev=2 straglev=4 scatlev=4 $
```

```
$nhs $
$nsc $
$nzh $
$nrh nrhist=0 $
$nem $
$ncv $
```

```
SECTION
```

```
REFP
2 .18 0. 0. 3 !typ reff t0 grad0 mode (3=const p 4=with acc)
```

```
DENS
LIH 1
BEGS
```

```
!=====
```

```
CELL !----- regular cell
1
.FALSE.
```

```
BSOL ! multipole field input
4. 56 .180 5 1 1 1 1 1 1 1 1 1 2.0 ! mode, file, momentum, order, 1/r switch, 10 scale-factors, 1/
```

```
!-----
```

```
REPEAT
1200
```

```
OUTPUT
SREGION ! define a region
```

```
.023598775 1 0.0001          ! length, 1 radial subregion, step
1 0. 2                        ! radial extent
NONE                          ! no associated field
0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
LIH VAC                       ! LiH material surrounded by vacuum
WEDGE
1 0.5 0.01 180 1.0 1.5 0. 0. 0. 0.      !angle RVERT ZVERT Azymuth Apx Apy
```

```

SREGION                                ! define a region
.05 1 0.001                          ! length, 1 radial subregion, step
1 0. 2                                ! radial extent
NONE                                  ! no associated field
0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
VAC                                  ! vacuum
CBLOCK                                ! cylindrical block geometry
0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.

```

```

SREGION                      ! define a region
.05 1 0.001                  ! length, 1 radial subregion, step
1 0. 2                        ! radial extent
NONE                          ! no associated field
0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
VAC                            ! vacuum
CBLOCK                        ! cylindrical block geometry
0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.

```

```

SREGION                      ! define a region
.05 1 0.001                  ! length, 1 radial subregion, step
1 0. 2                       ! radial extent
NONE                          ! no associated field
0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
VAC                           ! vacuum
CBLOCK                       ! cylindrical block geometry
0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.

```

```

SREGION                      ! define a region
.05 1 0.001                  ! length, 1 radial subregion, step
1 0. 2                       ! radial extent
NONE                         ! no associated field
0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
VAC                          ! vacuum
CBLOCK                       ! cylindrical block geometry
0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.

```

```

SREGION                      ! define a region
.05 1 0.001                  ! length, 1 radial subregion, step
1 0. 2                        ! radial extent
NONE                          ! no associated field
0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.

```

```

VAC                                ! vacuum
CBLOCK                            ! cylindrical block geometry
0. 0. 0. 0. 0. 0. 0. 0. 0. 0.

```

```

OUTPUT
SREGION                          ! define a region
.05 1 0.001                      ! length, 1 radial subregion, step
1 0. 2                          ! radial extent
NONE                             ! no associated field
0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
VAC                                ! vacuum
CBLOCK                            ! cylindrical block geometry
0. 0. 0. 0. 0. 0. 0. 0. 0. 0.

```

```

OUTPUT
SREGION                          ! define a region
.05 1 0.001                      ! length, 1 radial subregion, step
1 0. 2                          ! radial extent
NONE                             ! no associated field
0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
VAC                                ! vacuum
CBLOCK                            ! cylindrical block geometry
0. 0. 0. 0. 0. 0. 0. 0. 0. 0.

```

```

OUTPUT
SREGION                          ! define a region
.05 1 0.001                      ! length, 1 radial subregion, step
1 0. 2                          ! radial extent
NONE                             ! no associated field
0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
VAC                                ! vacuum
CBLOCK                            ! cylindrical block geometry
0. 0. 0. 0. 0. 0. 0. 0. 0. 0.

```

```

OUTPUT
SREGION                          ! define a region
.05 1 0.001                      ! length, 1 radial subregion, step
1 0. 2                          ! radial extent
NONE                             ! no associated field
0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
VAC                                ! vacuum
CBLOCK                            ! cylindrical block geometry
0. 0. 0. 0. 0. 0. 0. 0. 0. 0.

```

```

OUTPUT
SREGION                          ! define a region
.05 1 0.001                      ! length, 1 radial subregion, step
1 0. 2                          ! radial extent
NONE                             ! no associated field
0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
VAC                                ! vacuum
CBLOCK                            ! cylindrical block geometry
0. 0. 0. 0. 0. 0. 0. 0. 0. 0.

```

```

ENDREPEAT

ENDCELL
ENDSECTION

```

## D.2 input for003.dat beam file

This input file defines the initial positions and momenta of three input muons. The initial positions are set to zero with respect to the 0.5 m reference radius of the bent solenoid field. The initial momentum of the three muons is 0.18  $GeV/c$  in the  $z$  direction.

```

title
0 0 0 0 0 0 0 0 0
  1 0 2 0 0.0000E+00 1.0
0.00E-00 0.0000E+00 0.0000E+00 0.0000E-03 0.0000E+00 1.8000E-01 0 0 0 0 0 0
  2 0 2 0 0.0000E+00 1.0
0.00E-00 0.0000E+00 0.0000E+00 0.0000E-03 0.0000E+00 1.8000E-01 0 0 0 0 0 0
  3 0 2 0 0.0000E+00 1.0
0.00E-00 0.0000E+00 0.0000E+00 0.0000E-03 0.0000E+00 1.8000E-01 0 0 0 0 0 0

```

## D.3 input for056.dat field file

This file sets the parameters of the bent solenoid field. This field is sinusoidal with respect to the  $z$  coordinate (In ICOOL, the vertical component is  $y$  and  $z$  is the coordinate along the reference path.). The period is about 0.5236  $m$  corresponding to 1/6 the circumference of a circle with reference radius of 0.5  $m$ . The field is independent with respect to radius and depends on the  $z$  coordinate as  $B_y(z) = 1.168 - 1.651 \cos(2\pi z/0.5236)$  where the field is in Tesla and the distance is in meters.

```

test 6 sector anticyclotron
0.523598775 1 1 1 1 1 1 1 1 1 1 1 1 1
1
n      s      d      a      q
0 0 0 1.167764501 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0
1 0 0 -1.651468395 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0

```

## D.4 output for009.dat file

The following shows the kinematic information of the reference muon and the three muons through one wedge sregion and one vacuum sregion. Each data line wraps around to form three actual lines.

```

#test ring 180 MeV (H5b)
# units = [s] [m] [GeV/c] [T] [V/m]
# evt par typ flg reg time x y z
Px Py Pz Bx By Bz wt Ex Ey Ez arc
polX polY polZ
0 0 2 0 1 0.0E+00 0.0E+00 0.0E+00 0.0E+00

```

```

0.0E+00 0.0E+00 1.8E-01 0.0E+00 0.0E+00 0.0E+00 1 0 0 0 0.0E+00
0 0 0
1 0 2 0 1 0.0E+00 0.0E+00 0.0E+00 0.0E+00
0.0E+00 0.0E+00 1.8E-01 0.0E+00 0.0E+00 0.0E+00 1 0 0 0 0.0E+00
0 0 0
2 0 2 0 1 0.0E+00 0.0E+00 0.0E+00 0.0E+00
0.0E+00 0.0E+00 1.8E-01 0.0E+00 0.0E+00 0.0E+00 1 0 0 0 0.0E+00
0 0 0
3 0 2 0 1 0.0E+00 0.0E+00 0.0E+00 0.0E+00
0.0E+00 0.0E+00 1.8E-01 0.0E+00 0.0E+00 0.0E+00 1 0 0 0 0.0E+00
0 0 0
0 0 2 0 5 9.1E-11 0.0E+00 0.0E+00 2.4E-02
0.0E+00 0.0E+00 1.8E-01 0.0E+00 -4.2E-01 0.0E+00 1 0 0 0 2.4E-02
0 0 0
1 0 2 0 5 9.2E-11 7.8E-04 1.2E-04 2.4E-02
1.1E-02 1.9E-03 1.7E-01 2.4E-10 -4.2E-01 6.4E-04 1 0 0 0 2.4E-02
0 0 0
2 0 2 0 5 9.2E-11 1.1E-03 -7.4E-05 2.4E-02
1.5E-02 -1.7E-03 1.7E-01 -6.2E-11 -4.2E-01 -4.1E-04 1 0 0 0 2.4E-02
0 0 0
3 0 2 0 5 9.2E-11 7.8E-04 -1.6E-04 2.4E-02
1.2E-02 -1.5E-03 1.8E-01 -6.2E-10 -4.2E-01 -8.8E-04 1 0 0 0 2.4E-02
0 0 0

```

The orbit in the LAB  $(x, y)$  plane can be viewed with **Historoot** which can be obtained by downloading **G4Beamline** from [www.muonsinc.com](http://www.muonsinc.com). With **for009.dat** as the input file to **Historoot**, the orbit in the LAB  $(x, y)$  frame is obtained by plotting  $(x + 0.5 \text{ m}) \cos(z/0.5 \text{ m})$  vs.  $(x + 0.5 \text{ m}) \sin(z/0.5 \text{ m})$ . The radius is the LAB frame of the anticyclotron is  $(x + 0.5 \text{ m})$  where  $x$  is the the horizontal coordinate with respect to the ICOOL reference path which has a bent solenoid radius of curvature  $0.5 \text{ m}$ . Fig. 3 shows the orbits of three muons in this example ICOOL anticyclotron simulation which are stopped in about  $200 \text{ ns}$ .

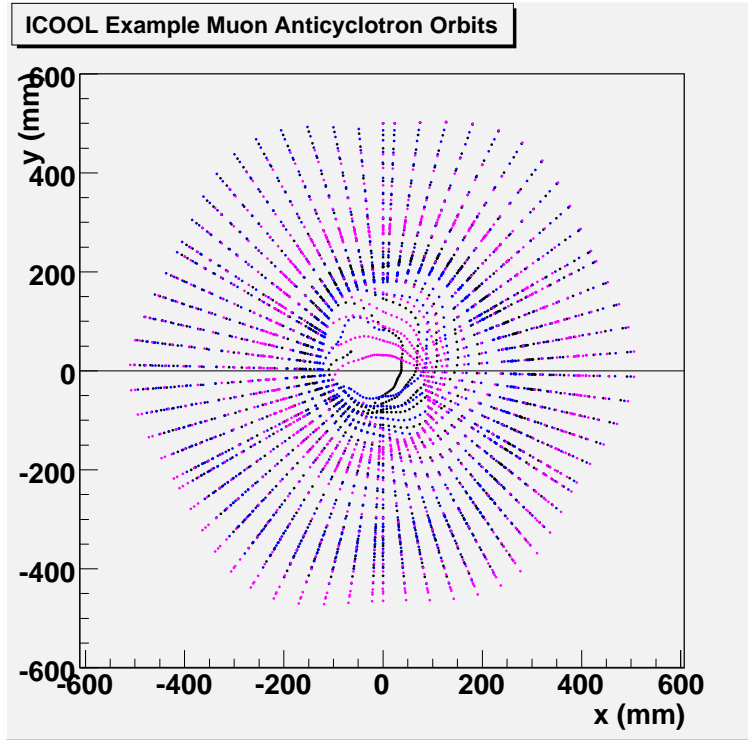


Figure 3: Orbits of three color coded muons in an ICOOL simulated anticyclotron. The output is generated by `Historoot`. Each point in the orbit is at the location of the end of an sregion. Unlike the G4Beamline simulation, the magnetic field in this ICOOL simulation is independent of radius and does not include an inner magnetic bottle.